The Nash bargaining solution:

This is referred to the *axiomatic* approach to bargaining. The reason is that Nash was interested in finding a *solution with particular properties* to a *bargaining problem*. Let us first define the *"problem"* and the properties that the *"solution"* must have. There are two agents[1], $j = 1, 2$, with utility functions u_j. There is an arbitrary set of outcomes A. D is the outcome in case the agents cannot reach an agreement (disagreement or threat point). Define $S = \{(u_1(a), u_2(a)), a \in A\}$ and $d = (d_1, d_2)$, where $d_j = u_j(D)$. Suppose that S is compact and convex and that $d \in S$. Also assume that $\exists s \in S$, such that $s_j > d_j$, $\forall j = 1, 2$.

$$\begin{aligned} & (S, d) \text{ is the bargaining problem} \\ f \quad : \quad & (S, d) \longrightarrow S \text{ is a solution to } (S, d) \end{aligned}$$

Nash was looking for a "solution" with the following properties:

- $(A1)$ *Invariance to utility choices:*

Given (S, d) and (S', d') defined by $s'_j = \alpha_j s_j + \beta_j$ and $d'_j = \alpha_j d_j + \beta_j$, then $f_j(S', d') = \alpha_j f_j(S, d) + \beta_j$

- $(A2)$ *Symmetry:*

If $d_1 = d_2$ and $(s_1, s_2) \in S \Longleftrightarrow (s_2, s_1) \in S$, then $f_1(S, d) = f_2(S, d)$

- $(A3)$ *Independence of irrelevant alternatives:*

If (S, d) and (S', d) satisfy $S \subset S'$ and $f(S', d) \in S$, then $f(S, d) = f(S', d)$

- $(A4)$ *Pareto efficiency:*

Given (S, d), if $s \in S$ and $s' \in S$ and $s'_j > s_j, \forall j = 1, 2$, then $f(S, d) \neq s$

Nash (1950) showed that the **unique** solution to this problem is[2]:

$$f(S, d) = \underset{s_1 \geq d_1, s_2 \geq d_2}{Arg \max} \; (s_1 - d_1)(s_2 - d_2) \tag{1}$$

To sketch the proof, it will be useful to draw a graph. By $(A1)$, one can choose the set of possible outcome S_1, such that $d = (0, 0)$ (normalization of the utility functions). Denote by S_2 the intersection of S_1 and the positive quadrant. Let $(u_1^*, u_2^*) = \underset{s \in S_2}{Arg \max} u_1 u_2$. By assumption, S_2 is non-empty, compact and convex, which guarantees existence of the maximizers. Uniqueness is obtained from the convexity assumption. By $(A1)$, choose u_1, u_2 such that $(u_1^*, u_2^*) = (u^*, u^*)$ lies on the 45^o line (normalization of the utility functions).

[1] In what follows, we are only interested in situations where **two** agents bargain.

[2] Remark that if requirement $(A2)$ were dropped, then there is a continuum of solutions:

$$f_\theta(S, d) = \underset{s_1 \geq d_1, s_2 \geq d_2}{Arg \max} \; (s_1 - d_1)^\theta (s_2 - d_2)^{(1-\theta)}$$

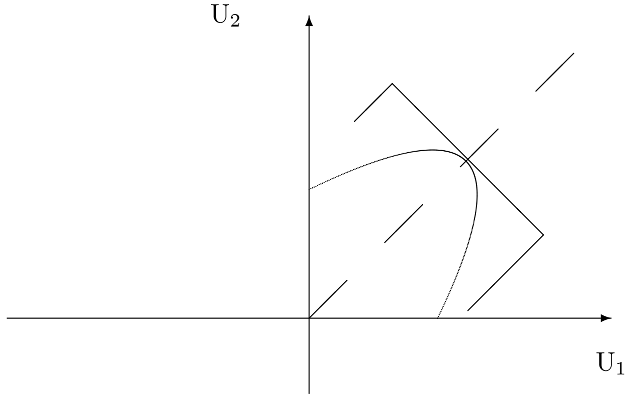

Figure 1: Graphical transformation used in Nash's proof

Notice that every point of S_2 is such that $u_1 + u_2 \leq 2u^*$[3]. Let B be a square, symmetric relative to the 45^o line, one side of which is supported by $u_1 + u_2 = 2u^*$, that includes S (of course, it is not unique). It exists since S is bounded. Then by $(A2)$, $f(B, O)$ is located on the 45^o line. By $(A4)$, $f(B, O) = (u^*, u^*)$. By $(A3)$, $f(S, O) = f(B, O)$. Hence, given the normalizations performed, $f(S, d)$ is located at (u^*, u^*). Remarkably, it can be proved that uniqueness of the bargaining solution cannot be obtained with a proper subset of these four axioms.

[3]Suppose that there exists a point $M = (u_1, u_2) \in S_2$ such that $u_1 + u_2 > 2u^*$. Then, there exists a point between M and $N = (u^*, u^*)$ that belongs to S, for which $u_1 u_2 > u^{*2}$ (by convexity of S_2).